A USER'S GUIDE TO CONTINUOUSLY CONTROLLED ALGEBRA

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ABSTRACT. Continuously controlled algebra is an important tool for proving the Farrell-Jones conjecture and the Novikov conjecture. The purpose of this expository article is to present an accessible introduction to continuously controlled algebra and the continuously controlled version of assembly maps.

1. Introduction

Controlled algebra was first considered by Connell and Hollingsworth in [12], and its first major applications were developed by Quinn in [24]. Continuously controlled algebra was introduced by Anderson, Connolly, Ferry and Pedersen [1] and was used by Carlsson and Pedersen [10, 11] to analyze assembly maps in algebraic K- and L-theory. Carlsson and Pedersen used this theory to prove the Novikov conjecture for a large class of groups by showing that the assembly map was a split injection. Motivated by the groundbreaking work of Farrell and Jones [14], Bartels, Lück and Reich [5] have recently used the continuously controlled version of the assembly map in their proof of the K-theoretic Farrell-Jones conjecture for word hyperbolic groups, and Bartels and Lück have announced the analogous result in L-theory. These two isomorphism results combine to prove the Borel conjecture for these groups.

The aim of this article is to provide a friendly introduction to continuously controlled algebra, hopefully making the general framework of this proof technique a little more accessible. To achieve this modest goal, we focus on the continuously controlled algebra approach to equivariant homology theories and assembly maps.

The objects of study in this theory are additive categories of so-called "geometric modules." The notation used in the literature varies from paper to paper and is often quite involved. We hope the presentation here will serve to demystify these categories and to

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provide the reader with an understanding of the underlying concepts. Continuously controlled categories are introduced without group actions and are illustrated with several examples. Group actions are then incorporated into the definition, so that an equivariant homology theory and corresponding assembly map can be defined. The presentation focuses on algebraic K-theory. To study algebraic L-theory one has to be careful with involutions, but the treatment is virtually the same. We conclude with a general discussion of how this theory has been used to prove isomorphism and split injectivity results.

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2. Continuously controlled algebra

Let us begin by recalling some categorical terminology. Two functors F and G between categories $\mathcal{A}$ and $\mathcal{B}$ are *naturally equivalent* if there is a natural transformation from F to G that is an isomorphism for every object in $\mathcal{A}$. Two categories $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if there are functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that FG is naturally equivalent to $\mathrm{id}_\mathcal{B}$ and GF is naturally equivalent to $\mathrm{id}_\mathcal{A}$.

An *additive category* $\mathcal{A}$ is a small category in which every hom-set (i.e., the set of morphisms between two objects) is an abelian group, morphism composition is bilinear, there is a zero object, and for every finite collection of objects $A_1, \ldots, A_n$ in $\mathcal{A}$, the biproduct $A_1 \oplus \cdots \oplus A_n$ is an object of $\mathcal{A}$ (a biproduct is both a product and a coproduct). An *additive functor* between additive categories $\mathcal{A}$ and $\mathcal{B}$ is a functor that is a group homomorphism for every hom-set in $\mathcal{A}$. Two additive categories are equivalent when they are equivalent by additive functors.

Let $\mathcal{A}$ be a small additive category, and let $\mathbb{K}^{-\infty}(\mathcal{A})$ denote the non-connective K-theory spectrum associated with the symmetric monoidal category obtained from $\mathcal{A}$ by restricting to isomorphisms. That is, $\pi_n(\mathbb{K}^{-\infty}(\mathcal{A})) \cong K_n(\mathcal{A})$ for every integer n. In particular, if $\mathcal{F}_R$ is the category of finitely generated free R-modules, then $\pi_n(\mathbb{K}^{-\infty}(\mathcal{F}_R)) \cong K_n(R)$ for every integer n; $\mathbb{K}^{-\infty}$ is a functor from the category of small additive categories to the category of spectra. For a more detailed description of this functor, see [22, 9].

Let X be a CW complex, $Y \subseteq X$ be a closed subcomplex, and $\mathcal{A}$ be a small additive category. A subset $K \subseteq X$ is called *relatively compact* if the closure of K in X is compact.

It is called *locally finite* if for every point x in K there is a neighborhood U of x such that $U \cap K = \{x\}$.

Definition 2.1. The continuously controlled category $\mathcal{B}(X, Y; \mathcal{A})$ is an additive category whose objects $A = (A_x)$ are collections of objects of $\mathcal{A}$ indexed by the points in the space $X - Y$ such that the *support of* A, $\mathrm{supp}(A) = \{x \in X - Y \mid A_x \neq 0\}$, is

(1) locally finite in $X - Y$; and
(2) relatively compact in X.

A morphism $\phi : A \to B$ is a collection $\phi = (\phi_x^{x'} : A_x \to B_{x'})$ of morphisms in $\mathcal{A}$ such that

(1) for every $x \in X - Y$, the set $\{x' \in X - Y \mid \phi_x^{x'} \neq 0 \text{ or } \phi_{x'}^{x} \neq 0\}$ is finite; and
(2) $\phi : A \to B$ is *continuously controlled* at Y. That is, for every $y \in Y$ and every neighborhood $U \subseteq X$ of y, there is a neighborhood $V \subseteq X$ of y such that $\phi_x^{x'} = 0$ and $\phi_{x'}^{x} = 0$ whenever $x \in V$ and $x' \notin U$.

The continuously controlled category depends functorially on the CW pair (X, Y) (for a proof see, for example, [4, Section 3.3]). A map $f : (X, Y) \to (X', Y')$ induces a functor $f_* : \mathcal{B}(X, Y; \mathcal{A}) \to \mathcal{B}(X', Y'; \mathcal{A})$, defined by $f_*(A)_{x'} = \bigoplus_{x \in f^{-1}(x')} A_x$. Given a morphism $\phi : A \to B$ in $\mathcal{B}(X, Y; \mathcal{A})$, $f_*(\phi) : f_*(A) \to f_*(B)$ is defined in the obvious way.

Example 2.2 (Forget-control)**.** Consider the categories $\mathcal{B}(X \times [0,1], X \times \{1\}; \mathcal{A})$ and $\mathcal{B}(CX, *; \mathcal{A})$, where $CX = X \times [0,1]/X \times \{1\}$ is the cone on X and $*$ is the cone point. The control spaces for these two categories are the same, namely $X \times [0,1)$, but the control conditions are different. In $\mathcal{B}(X \times [0,1], X \times \{1\}; \mathcal{A})$ the continuous control condition on morphisms is defined with respect to the boundary $X \times \{1\}$, whereas in $\mathcal{B}(CX, *; \mathcal{A})$ the continuous control condition is defined with respect to the one-point boundary. Therefore, the continuous control condition on morphisms in $\mathcal{B}(X \times [0,1], X \times \{1\}; \mathcal{A})$ is more restrictive than the one in $\mathcal{B}(CX, *; \mathcal{A})$. The quotient map $X \times [0,1] \to X \times [0,1]/X \times \{1\}$, induces a functor $\mathcal{B}(X \times [0,1], X \times \{1\}; \mathcal{A}) \to \mathcal{B}(CX, *; \mathcal{A})$ that is essentially the identity on objects and morphisms. For this reason, it is known as a *forget-control functor*. In addition, because of the relatively compact condition on objects, the map $CX \to \mathrm{pt} \times [0,1]$, which collapses X to a point, induces an equivalence of categories $\mathcal{B}(CX, *; \mathcal{A}) \cong \mathcal{B}([0,1], \{1\}; \mathcal{A})$. This helps to illustrate the difference between $\mathcal{B}(X \times [0,1], X \times \{1\}; \mathcal{A})$ and $\mathcal{B}(CX, *; \mathcal{A})$. (Compare with Example 2.3 below.)

An additive category $\mathcal{B}$ is called *flasque* if it admits an endofunctor $\Sigma : \mathcal{B} \to \mathcal{B}$ and a natural equivalence between $\mathrm{id}_{\mathcal{B}} \oplus \Sigma$ and Σ. By the Additivity Theorem, such a category has trivial K-theory. Using an argument of this type to prove that a category has trivial K-theory is referred to as an *Eilenberg swindle*.

Example 2.3. We will use an Eilenberg swindle to show that $\mathcal{B}([0,1],\{1\};\mathcal{A})$ has trivial K-theory. The idea for constructing the necessary endofunctor is to shift the objects and morphisms towards 1. Let A and B be objects of $\mathcal{B}([0,1],\{1\};\mathcal{A})$, and let $\phi : A \to B$ be a morphism. Let $S : \mathcal{B}([0,1],\{1\};\mathcal{A}) \to \mathcal{B}([0,1],\{1\};\mathcal{A})$ be the endofunctor defined by

$$S(A)_t := A_{2t-1},$$
$$S(\phi)_s^t := \phi_{2s-1}^{2t-1} : A_{2s-1} \to B_{2t-1}.$$

Now define the endofunctor $\Sigma : \mathcal{B}([0,1],\{1\};\mathcal{A}) \to \mathcal{B}([0,1],\{1\};\mathcal{A})$ by

$$\Sigma(A) := \bigoplus_{n\geq 1} S^n(A),$$
$$\Sigma(\phi) := \bigoplus_{n\geq 1} S^n(\phi).$$

For each object A there is a continuously controlled isomorphism $U_A : A \to S(A)$, defined by

$$(U_A)_s^t := \begin{cases} \mathrm{id}_{A_s} & \text{if } s = 2t-1 \\ 0 & \text{otherwise} \end{cases}.$$

The desired natural equivalence, $\eta : \mathrm{id} \oplus \Sigma \to \Sigma$, is given by $\eta(A) := \bigoplus_{n\geq 0} U_{S^n(A)}$. Therefore, $\mathbb{K}^{-\infty}(\mathcal{B}([0,1],\{1\};\mathcal{A}))$ is weakly contractible (i.e., it has trivial homotopy groups).

At first glance it appears that $\mathcal{B}(X \times [0,1], X \times \{1\};\mathcal{A})$ admits an Eilenberg swindle for any X by sliding the objects and morphisms towards 1, as was done in Example 2.3. But this does not work in general. Remember that the continuous control condition on morphisms says that the components of a morphism must become shorter and shorter as they approach 1. This must happen in the X-direction, as well as in the $[0,1)$-direction. For example, consider the two-point space $S^0 = \{-1,1\}$. By the continuous control condition, the components of a morphism in $\mathcal{B}(S^0 \times [0,1], S^0 \times \{1\};\mathcal{A})$ are zero between $(-1,s)$ and $(1,t)$, if s or t is sufficiently close to 1. Therefore, a morphism with a non-zero component between $(-1,s)$ and $(1,t)$ for some s and t away from 1, cannot be shifted arbitrarily close to $S^0 \times \{1\}$.

The *support at infinity* of an object A in $\mathcal{B}(X,Y;\mathcal{A})$ is the set of limit points of $\operatorname{supp}(A)$ in Y.

Definition 2.4. Let C be a closed subset of Y. The category $\mathcal{B}(X,Y;\mathcal{A})_C$ is the full subcategory of $\mathcal{B}(X,Y;\mathcal{A})$ on objects whose support at infinity is contained in C.

Example 2.5. Consider $\mathcal{B}(X\times[0,1],X\times\{1\};\mathcal{A})_\emptyset$, the full subcategory of $\mathcal{B}(X\times[0,1],X\times\{1\};\mathcal{A})$ whose objects have empty support at infinity. Because of the relatively compact and locally finite conditions, the support of every object is finite. Furthermore, the continuous control condition is vacuous. Therefore, $\mathcal{B}(X\times[0,1],X\times\{1\};\mathcal{A})_\emptyset$ and $\mathcal{A}$ are equivalent additive categories.

Definition 2.6. Let W be an open subset of Y. The *germ category* $\mathcal{B}(X,Y;\mathcal{A})^W$ has the same objects as $\mathcal{B}(X,Y;\mathcal{A})$, but morphisms are identified if they agree in a neighborhood of W. Specifically, $\phi,\psi : A\to B$ are identified if there is a neighborhood $U\subseteq X$ of W such that $x\in U$ or $x'\in U$ implies that $\phi_x^{x'}=\psi_x^{x'}$.

Germ categories are very interesting. As noted above, $\mathcal{B}(S^0\times[0,1],S^0\times\{1\};\mathcal{A})$ is not equivalent to $\mathcal{B}([0,1],\{1\};\mathcal{A})\oplus\mathcal{B}([0,1],\{1\};\mathcal{A})$, since $\mathcal{B}([0,1],\{1\};\mathcal{A})\oplus\mathcal{B}([0,1],\{1\};\mathcal{A})$ is flasque (because each summand is flasque), while $\mathcal{B}(S^0\times[0,1],S^0\times\{1\};\mathcal{A})$ is not. However, it is true that the germ categories $\mathcal{B}(S^0\times[0,1],S^0\times\{1\};\mathcal{A})^{S^0\times\{1\}}$ and $\mathcal{B}([0,1],\{1\};\mathcal{A})^{\{1\}}\oplus\mathcal{B}([0,1],\{1\};\mathcal{A})^{\{1\}}$ are equivalent. The reason is that we are taking germs at 1, which implies that every morphism has a representative that does not "jump between levels." What is meant by this is the following. Let ϕ be a morphism in $\mathcal{B}(S^0\times[0,1],S^0\times\{1\};\mathcal{A})$. The continuous control condition implies that there is a neighborhood $V\subseteq S^0\times[0,1]$ of the point $(1,1)$ such that $\phi_{(1,t)}^{(-1,s)}=0=\phi_{(-1,s)}^{(1,t)}$ whenever $(1,t)$ is in V. Now define a morphism ψ such that $\psi_{(1,t)}^{(-1,s)}=0=\psi_{(-1,s)}^{(1,t)}$ for all s and t, and $\psi_x^{x'}=\phi_x^{x'}$ otherwise. Therefore, ϕ and ψ represent the same morphism in the germ category $\mathcal{B}(S^0\times[0,1],S^0\times\{1\};\mathcal{A})^{S^0\times\{1\}}$, and the components of ψ are zero between points in $\{-1\}\times[0,1)$ and $\{1\}\times[0,1)$. More generally, we have:

Lemma 2.7. *If T is a discrete space, then*

$$\mathcal{B}(T\times[0,1],T\times\{1\};\mathcal{A})^{T\times\{1\}}\cong\bigoplus_T\mathcal{B}([0,1],\{1\};\mathcal{A})^{\{1\}}.$$

Sketch of proof. As with the two point space, since we are taking germs at 1, every morphism has a representative that does not jump between levels. The reason for the direct sum is the relative compactness condition, which implies that each object in $\mathcal{B}(T\times[0,1], T\times\{1\}; \mathcal{A})^{T\times\{1\}}$ can only be non-zero for finitely many points in T. □

The categories from Definitions 2.4 and 2.6 have a special relationship. In [20], Karoubi introduced the following notion.

Definition 2.8. Let $\mathcal{A}$ be a full subcategory of an additive category $\mathcal{U}$. Denote the objects of $\mathcal{A}$ by the letters A through F and the objects of $\mathcal{U}$ by the letters U through W. Then $\mathcal{U}$ is said to be $\mathcal{A}$-*filtered* if every object U has a family of decompositions $\{U = E_\alpha \oplus U_\alpha\}$ where

(i) the decomposition forms a filtered poset under the partial order in which $E_\alpha \oplus U_\alpha \leq E_\beta \oplus U_\beta$ whenever $U_\beta \subseteq U_\alpha$ and $E_\alpha \subseteq E_\beta$;
(ii) every map $A \to U$ factors through E_α for some α;
(iii) every map $U \to A$ factors through E_α for some α;
(iv) for each U and V, the filtration on $U \oplus V$ is equivalent to the sum of the filtrations $\{U = E_\alpha \oplus U_\alpha\}$ and $\{V = F_\beta \oplus V_\beta\}$; that is, $\{U \oplus V = (E_\alpha \oplus F_\beta) \oplus (U_\alpha \oplus V_\beta)\}$.

Karoubi also defined the *quotient category* $\mathcal{U}/\mathcal{A}$ to have the same objects as $\mathcal{U}$, but morphisms ϕ, $\psi : U \to V$ are identified if their difference, $\phi - \psi$, factors through $\mathcal{A}$. The key fact is that the induced sequence

$$\mathbb{K}^{-\infty}(\mathcal{A}) \to \mathbb{K}^{-\infty}(\mathcal{U}) \to \mathbb{K}^{-\infty}(\mathcal{U}/\mathcal{A})$$

is a homotopy fibration of spectra, which yields a long exact sequence of homotopy groups. For this reason Karoubi filtrations play a major role in the subject.

It is straightforward to check that

$$\mathcal{B}(X, Y; \mathcal{A})_C \to \mathcal{B}(X, Y; \mathcal{A})$$

is a Karoubi filtration, where C is a closed subset of Y. Furthermore, the corresponding quotient category is precisely the germ category $\mathcal{B}(X, Y; \mathcal{A})^{Y-C}$. Therefore, the sequence

$$\mathcal{B}(X, Y; \mathcal{A})_C \to \mathcal{B}(X, Y; \mathcal{A}) \to \mathcal{B}(X, Y; \mathcal{A})^{Y-C} \tag{1}$$

yields a homotopy fibration of spectra after applying $\mathbb{K}^{-\infty}$. Similarly, it is an exercise to show that

$$\mathcal{B}(X,Y;\mathcal{A})^W_C \to \mathcal{B}(X,Y;\mathcal{A})^W \to \mathcal{B}(X,Y;\mathcal{A})^{W-C} \tag{2}$$

is a Karoubi filtration sequence, where W is an open subset of Y.

Example 2.9. An instance of (1) that is useful for studying assembly maps is the sequence

$$\mathcal{B}(X\times[0,1],X\times\{1\};\mathcal{A})_\emptyset \to \mathcal{B}(X\times[0,1],X\times\{1\};\mathcal{A}) \to \mathcal{B}(X\times[0,1],X\times\{1\};\mathcal{A})^{X\times\{1\}}.$$

When X is a point, the associated long exact sequence implies that

$$K_{n+1}(\mathcal{B}([0,1],\{1\};\mathcal{A})^{\{1\}}) \cong K_n(\mathcal{B}([0,1],\{1\};\mathcal{A})_\emptyset) \cong K_n(\mathcal{A}),$$

since $\mathbb{K}^{-\infty}(\mathcal{B}([0,1],\{1\};\mathcal{A}))$ is weakly contractible. Therefore, $\Omega\mathbb{K}^{-\infty}(\mathcal{B}([0,1],\{1\};\mathcal{A})^{\{1\}})$ is weakly homotopy equivalent to $\mathbb{K}^{-\infty}(\mathcal{A})$.

The functor $\Omega\mathbb{K}^{-\infty}(\mathcal{B}(-\times[0,1],-\times\{1\};\mathcal{A})^{-\times\{1\}})$ from the category of CW complexes to the category of spectra is homotopy invariant and excisive, thereby yielding the generalized homology theory $H_*(X;\mathbf{K}_\mathcal{A}) := \pi_*(\Omega\mathbb{K}^{-\infty}(\mathcal{B}(-\times[0,1],-\times\{1\};\mathcal{A})^{-\times\{1\}}))$. By Example 2.9, its value at a point is the K-theory of $\mathcal{A}$. Karoubi filtrations are needed to establish the fact that $H_*(X;\mathbf{K}_\mathcal{A})$ is in fact an homology theory. Excision is proved by showing that the pushout diagram of CW complexes

$$\begin{tikzcd} X\cap Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X\cup Y \end{tikzcd}$$

induces a homotopy Cartesian square

$$\begin{tikzcd} \mathbb{K}^{-\infty}(\mathcal{B}(X\cap Y;\mathcal{A})^{(X\cap Y)\times\{1\}}) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(\mathcal{B}(X;\mathcal{A})^{X\times\{1\}}) \arrow[d] \\ \mathbb{K}^{-\infty}(\mathcal{B}(Y;\mathcal{A})^{Y\times\{1\}}) \arrow[r] & \mathbb{K}^{-\infty}(\mathcal{B}(X\cup Y;\mathcal{A})^{(X\cup Y)\times\{1\}}) \end{tikzcd} \tag{3}$$

where $\mathcal{B}(-\times[0,1],-\times\{1\};\mathcal{A})$ is denoted by $\mathcal{B}(-;\mathcal{A})$. To prove this, consider the following commutative diagram of Karoubi filtrations induced by inclusion.

$$\begin{tikzcd} \mathcal{B}(X;\mathcal{A})^{X\times\{1\}}_{(X\cap Y)\times\{1\}} \arrow[r] \arrow[d] & \mathcal{B}(X;\mathcal{A})^{X\times\{1\}} \arrow[r] \arrow[d] & \mathcal{B}(X;\mathcal{A})^{(X-Y)\times\{1\}} \arrow[d] \\ \mathcal{B}(X\cup Y;\mathcal{A})^{(X\cup Y)\times\{1\}}_{Y\times\{1\}} \arrow[r] & \mathcal{B}(X\cup Y;\mathcal{A})^{(X\cup Y)\times\{1\}} \arrow[r] & \mathcal{B}(X\cup Y;\mathcal{A})^{(X-Y)\times\{1\}} \end{tikzcd} \tag{4}$$

It is an exercise to show that, when C is a closed subset of X, the inclusion functor $\mathcal{B}(C;\mathcal{A})^{C\times\{1\}} \to \mathcal{B}(X;\mathcal{A})^{X\times\{1\}}_{C}$ produces an equivalence of categories. Therefore, $\mathcal{B}(X\cap Y;\mathcal{A})^{(X\cap Y)\times\{1\}} \cong \mathcal{B}(X;\mathcal{A})^{X\times\{1\}}_{(X\cap Y)\times\{1\}}$ and $\mathcal{B}(Y;\mathcal{A})^{Y\times\{1\}} \cong \mathcal{B}(X\cup Y;\mathcal{A})^{(X\cup Y)\times\{1\}}_{Y\times\{1\}}$. One also checks that $\mathcal{B}(X;\mathcal{A})^{(X-Y)\times\{1\}} \to \mathcal{B}(X\cup Y;\mathcal{A})^{(X-Y)\times\{1\}}$ has an inverse induced by projection. Thus, diagram (4) becomes

$$\begin{tikzcd} \mathcal{B}(X\cap Y;\mathcal{A})^{(X\cap Y)\times\{1\}} \arrow[r] \arrow[d] & \mathcal{B}(X;\mathcal{A})^{X\times\{1\}} \arrow[r] \arrow[d] & \mathcal{B}(X;\mathcal{A})^{(X-Y)\times\{1\}} \arrow[d, "\cong"] \\ \mathcal{B}(Y;\mathcal{A})^{Y\times\{1\}} \arrow[r] & \mathcal{B}(X\cup Y;\mathcal{A})^{(X\cup Y)\times\{1\}} \arrow[r] & \mathcal{B}(X\cup Y;\mathcal{A})^{(X-Y)\times\{1\}} \end{tikzcd} \tag{5}$$

Applying $\mathbb{K}^{-\infty}$ to (5) yields a commutative diagram of spectra in which each row is a fibration. Since fibration sequences are also cofibration sequences in the category of spectra, the two rows of the induced diagram are cofibration sequences with equivalent cofibers. Therefore, diagram (3) is a homotopy co-Cartesian square, which means it is a homotopy Cartesian square, since we are working in the category of spectra.

The above argument can be found in [17, Corollary 9.3], [15, p. 48] and [4, Proposition 4.3]. For a proof of homotopy invariance, the reader is referred to [4, Section 5]. The proof employs a trick, dating back to Pedersen and Weibel [22], that makes clever use of Eilenberg swindles.

3. Equivariant homology and the assembly map

Given a discrete group G and a ring R, the classical assembly map in algebraic K-theory, introduced by Loday [21], is a map that relates the K-theory of the group ring $R[G]$ to the homology of BG with coefficients in the K-theory spectrum of R. In the revolutionary work of Farrell and Jones [14], a more general assembly map relating the K-theory of $R[G]$ to a generalized G-equivariant homology theory was formulated using Quinn's "homology of simplicially stratified fibrations" [24]. Davis and Lück [13] used the "orbit category"

to create an abstract approach to Farrell-Jones assembly, and Hambleton and Pedersen formalized the continuously controlled version of the Farrell-Jones assembly map in [15]. Recently, Bartels and Reich [7] used the Davis-Lück machinery to construct an assembly map with coefficients in an additive category on which G acts. This more general formulation allows one to study, for example, twisted group rings. It also possesses interesting inheritance properties (see [7, 16]). In this section the assembly map *with coefficients* is defined using continuously controlled algebra. In order to do this the corresponding equivariant homology theory must be developed.

An *additive category with right G-action* $\mathcal{A}$ is an additive category together with a collection of covariant functors $\{g^*\colon \mathcal{A} \to \mathcal{A} \mid g \in G\}$ such that $(g \circ h)^* = h^* \circ g^*$ and $e^* = \mathrm{id}_{\mathcal{A}}$. A subset $S \subseteq X$ is called *G-compact* if $S = G \cdot K$ for some compact subset $K \subseteq X$. It is called *relatively G-compact* if its closure in X is G-compact.

Definition 3.1. Let G be a discrete group, $\mathcal{A}$ be an additive category with right G-action, X be a G-CW complex and $Y \subseteq X$ be a closed G-invariant subcomplex of X. The category $\mathcal{D}(X, Y; \mathcal{A})$ has objects $A = (A_z)$, consisting of collections of objects of $\mathcal{A}$ indexed by $G \times (X - Y)$ such that the support of A, $\mathrm{supp}(A) = \{z \in G \times (X - Y) \mid A_z \neq 0\}$, is:

(1) locally finite in $G \times (X - Y)$; and
(2) relatively G-compact in $G \times X$.

A morphism $\phi : A \to B$ is a collection $\phi = (\phi_z^{z'} : A_z \to B_{z'})$ of morphisms in $\mathcal{A}$ such that:

(1) for every $z \in G \times (X - Y)$, the set $\{z' \in G \times (X - Y) \mid \phi_z^{z'} \neq 0 \text{ or } \phi_{z'}^{z} \neq 0\}$ is finite; and
(2) $\phi : A \to B$ is continuously controlled at Y, which means that for every $y \in Y$ and every G_y-invariant neighborhood $U \subseteq X$ of y, there is a G_y-invariant neighborhood $V \subseteq X$ of y such that $\phi_z^{z'} = 0$ and $\phi_{z'}^{z} = 0$ whenever $z \in G \times V$ and $z' \notin G \times U$.

Adding a factor of G to the control space, an idea due to Pedersen, creates an interesting action of G on $\mathcal{D}(X, Y; \mathcal{A})$. The right G-action on $\mathcal{D}(X, Y; \mathcal{A})$ is induced by the diagonal action of G on $G \times X$ and the right G-action on $\mathcal{A}$. It is given by

$$(g^*A)_z := g^*(A_{gz}),$$
$$(g^*\phi)_z^{z'} := g^*(\phi_{gz}^{gz'}).$$

In the corresponding fixed point category, $\mathcal{D}^G(X,Y;\mathcal{A})$, every object A and every morphism ϕ satisfy $A_z = (g^*A)_z = g^*(A_{gz})$ and $\phi_z^{z'} = (g^*\phi)_z^{z'} = g^*(\phi_{gz}^{gz'})$ for every g in G and every z, z' in $G \times (X - Y)$. If $\mathcal{A} = \mathcal{F}_R$ with the trivial G-action, then the objects of $\mathcal{D}^G(X \times [0,1); \mathcal{F}_R)$ are free $R[G]$-modules and the morphisms are $R[G]$-homomorphisms. Furthermore, the direct sum of the pieces of an object A over an entire orbit, $\bigoplus_{g' \in G} A_{g'z} = \bigoplus_{g \in G} A_{(g,gx)}$ for some x in $X - Y$, is a finitely generated free $R[G_x]$-module. Thus, we can think of objects in $\mathcal{D}^G(X,Y;\mathcal{F}_R)$ as being built out of finitely generated free $R[G_x]$-modules, where x is a point in X. In particular, notice that $\mathcal{D}^G(\mathrm{pt}, \emptyset; \mathcal{F}_R)$ is equivalent to $\mathcal{F}_{R[G]}$, the category of finitely generated free $R[G]$-modules. For this reason the category $\mathcal{D}^G(\mathrm{pt}, \emptyset; \mathcal{A})$ is denoted by $\mathcal{A}[G]$.

As in the unequivariant case, $\mathcal{D}^G(X,Y;\mathcal{A})$ depends functorially on the G-CW pair (X,Y) [4, Section 3.3]. In order to construct an equivariant homology theory and an assembly map, we will work with the category $\mathcal{D}^G(X \times [0,1], X \times \{1\}; \mathcal{A})$, where X is a G-CW complex. For brevity, denote this category by $\mathcal{D}^G(X \times [0,1); \mathcal{A})$.

Let $\mathcal{D}^G(X \times [0,1); \mathcal{A})_\emptyset$ be the full subcategory of $\mathcal{D}^G(X \times [0,1); \mathcal{A})$ on objects A such that the intersection of $X \times \{1\}$ with the closure of $\mathrm{supp}(A)$ in $X \times [0,1]$ is the empty set. Notice that, as in Example 2.9, the quotient category, which we denote by $\mathcal{D}^G(X \times [0,1); \mathcal{A})^{>0}$, is a germ category. The objects are the same as in $\mathcal{D}^G(X \times [0,1); \mathcal{A})$ but morphisms are identified if they agree close to $G \times X \times \{1\}$, i.e., on the complement of a neighborhood of $G \times X \times \{0\}$. (It is not difficult to check that taking germs and taking fixed categories commute.) Furthermore, there is a corresponding Karoubi filtration sequence

$$\mathcal{D}^G(X \times [0,1); \mathcal{A})_\emptyset \to \mathcal{D}^G(X \times [0,1); \mathcal{A}) \to \mathcal{D}^G(X \times [0,1); \mathcal{A})^{>0}.$$

Example 3.2. Let A be an object in $\mathcal{D}^G(\mathrm{pt} \times [0,1); \mathcal{F}_R)_\emptyset$. Since objects have empty support at infinity, the local finiteness condition implies that there are only finitely many $t \in [0,1)$ with $A_{(e,t)} \neq 0$, where e is the identity of G. By equivariance, $A_{(g,t)} = A_{(e,t)}$ for every $t \in [0,1)$. Therefore, A is a finitely generated free $R[G]$-module. Since the continuous control condition is vacuous, equivariance tells us that the morphisms in $\mathcal{D}^G(\mathrm{pt} \times [0,1); \mathcal{F}_R)_\emptyset$ are just $R[G]$-homomorphisms. Thus, $\mathcal{D}^G(\mathrm{pt} \times [0,1); \mathcal{F}_R)_\emptyset$ is equivalent to $\mathcal{F}_{R[G]}$. (In fact, $\mathcal{D}^G(X \times [0,1); \mathcal{F}_R)_\emptyset$ is equivalent to $\mathcal{F}_{R[G]}$ for every X, because of the relatively G-compact condition on objects.) As in Example 2.9, since $\mathcal{D}^G(\mathrm{pt} \times [0,1); \mathcal{F}_R)$ is flasque, $\Omega\mathbb{K}^{-\infty}(\mathcal{D}^G(\mathrm{pt} \times [0,1); \mathcal{F}_R)^{>0})$ is weakly homotopy equivalent to $\mathbb{K}^{-\infty}(\mathcal{F}_{R[G]})$.

More generally, we have the following lemma about orbits G/H.

Lemma 3.3. $\Omega\mathbb{K}^{-\infty}(\mathcal{D}^G(G/H \times [0,1); \mathcal{A})^{>0}) \simeq \mathbb{K}^{-\infty}(\mathcal{A}[H])$.

Proof. Equivariance tells us that the objects of $\mathcal{D}^G(G/H \times [0,1); \mathcal{A})^{>0}$ are determined by their value over the points in $\{eH\} \times [0,1)$. Since the isotropy at every point in $\{eH\} \times [0,1)$ is H, the objects over $\{eH\} \times [0,1)$ are objects in $\mathcal{A}[H]$. Since G/H is a discrete space and we are taking germs at 1, the morphisms in $\mathcal{D}^G(G/H \times [0,1); \mathcal{A})_\emptyset$ cannot jump between different points of G/H (compare with Lemma 2.7). Combining this with the equivariance of morphisms, we see that $\mathcal{D}^G(G/H \times [0,1); \mathcal{A})^{>0}$ is equivalent to $\mathcal{D}^{\{e\}}(\mathrm{pt} \times [0,1); \mathcal{A}[H])^{>0}$. Since $\mathcal{D}^{\{e\}}(\mathrm{pt} \times [0,1); \mathcal{A}[H])$ is flasque, $\Omega\mathbb{K}^{-\infty}(\mathcal{D}^G(G/H \times [0,1); \mathcal{A})^{>0}) \simeq \mathbb{K}^{-\infty}(\mathcal{A}[H])$. $\square$

The functor $\mathbf{K}(-; \mathcal{A})^G := \Omega\mathbb{K}^{-\infty}(\mathcal{D}^G(- \times [0,1); \mathcal{A})^{>0})$ from the category of G-CW complexes to the category of spectra is G-homotopy invariant and G-excisive [4, Section 5]. As in the unequivariant case, the proof of this fact makes use of Karoubi filtrations and Eilenberg swindles. In this sense $\mathbf{K}(-; \mathcal{A})^G$ defines a G-equivariant homology theory. The corresponding equivariant homology groups are denoted by $H^G_*(X; \mathbf{K}_\mathcal{A}) := \pi_*(\mathbf{K}(X; \mathcal{A})^G)$.

The collapse map, $X \to \mathrm{pt}$, induces the *continuously controlled assembly map*

$$A^G : \mathbf{K}(X; \mathcal{A})^G \to \mathbf{K}(\mathrm{pt}; \mathcal{A})^G,$$

which yields a map on homology, $A^G_* : H^G_*(X; \mathbf{K}_\mathcal{A}) \to H^G_*(\mathrm{pt}; \mathbf{K}_\mathcal{A}) = K_*(\mathcal{A}[G])$. In light of Example 2.2, this construction interprets assembly as a forget-control map. It is also worth noting that the continuously controlled assembly map is a map of fixed point spectra, since $\mathbb{K}^{-\infty}$ commutes with taking fixed sets. That is, if G acts on an additive category $\mathcal{B}$, then there is an induced action of G on $\mathbb{K}^{-\infty}(\mathcal{B})$, and the fixed point set of this spectrum, $\mathbb{K}^{-\infty}(\mathcal{B})^G$, is equivalent to the spectrum $\mathbb{K}^{-\infty}(\mathcal{B}^G)$. This implies that A^G is the map induced by the equivariant map of spectra $\mathbf{K}(X; \mathcal{A}) \to \mathbf{K}(\mathrm{pt}; \mathcal{A})$.

The continuously controlled version of the assembly map is homotopy equivalent to the one defined by Bartels and Reich, who used the Davis-Lück machinery for constructing assembly maps [7, 16]. In the case $\mathcal{A} = \mathcal{F}_R$, Hambleton and Pedersen used Lemma 3.3 to identify the Davis-Lück assembly map with the continuously controlled assembly map [15]. They also showed that both versions were equivalent to the classical definition of the assembly map.

The *Farrell-Jones conjecture with coefficients* predicts that the assembly map is an isomorphism for all coefficients $\mathcal{A}$ when $X = \mathrm{E}_{\mathcal{VC}yc}G$, where $\mathcal{VC}yc$ is the family of virtually

cyclic subgroups of G. [1] This is a very strong statement. For example, if the Farrell-Jones conjecture with coefficients is true for a group G, then the Fibered Farrell-Jones conjecture is true for G [7]. Furthermore, it would imply that the Farrell-Jones conjecture with coefficients is true for every subgroup of G. It also behaves well with respect to extensions (see [16]).

By the universal property, there is a map $\mathrm{E}_{\mathcal{F}in}G \to \mathrm{E}_{\mathcal{VC}yc}G$, where $\mathcal{F}in$ is the family of finite subgroups of G. Bartels [2] proved that for every discrete group G, the induced map on homology, $H_*^G(\mathrm{E}_{\mathcal{F}in}G;\mathbf{K}_{\mathcal{A}}) \to H_*^G(\mathrm{E}_{\mathcal{VC}yc}G;\mathbf{K}_{\mathcal{A}})$, is a split injection. (He showed this for $\mathcal{A} = \mathcal{F}_R$, but the proof works in the general case as well.) As a result, the Farrell-Jones conjecture implies that the assembly map for the family of finite subgroups, $H_*^G(\mathrm{E}_{\mathcal{F}in}G;\mathbf{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$, should be a split injection for every discrete group G.

4. Proving isomorphism and injectivity results

In this section we present an alternate description of the continuously controlled assembly map that is useful for proofs.

Consider the following commutative diagram.

$$\begin{tikzcd}
\mathcal{D}^G(X\times[0,1);\mathcal{A})_\emptyset \arrow[r] \arrow[d, "\cong"] & \mathcal{D}^G(X\times[0,1);\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{D}^G(X\times[0,1);\mathcal{A})^{>0} \arrow[d, "A^G"] \\
\mathcal{D}^G(\mathrm{pt}\times[0,1);\mathcal{A})_\emptyset \arrow[r] & \mathcal{D}^G(\mathrm{pt}\times[0,1);\mathcal{A}) \arrow[r] & \mathcal{D}^G(\mathrm{pt}\times[0,1);\mathcal{A})^{>0}
\end{tikzcd}$$

Since $\mathcal{D}^G(\mathrm{pt}\times[0,1);\mathcal{A})$ is flasque, $\mathbb{K}^{-\infty}(\mathcal{D}^G(\mathrm{pt}\times[0,1);\mathcal{A}))$ is weakly contractible. Therefore, it is a simple diagram chase to verify that the assembly map, A^G, is homotopy equivalent to the connecting map $\Omega\mathbb{K}^{-\infty}(\mathcal{D}^G(X\times[0,1);\mathcal{A})^{>0}) \to \mathbb{K}^{-\infty}(\mathcal{D}^G(X\times[0,1);\mathcal{A})_\emptyset)$.

This variant of the continuously controlled assembly map shows us that in order to prove isomorphism results one needs to prove that the category $\mathcal{D}^G(X\times[0,1);\mathcal{A})$ has trivial K-theory. For this reason, Bartels-Lück-Reich call this category *the obstruction category*. In [5] they were able to prove the K-theoretic Farrell-Jones conjecture for all word hyperbolic groups and all coefficient categories by showing that the K-theory of this

[1] Recall that if $\mathcal{F}$ is a family of subgroups of G that is closed under conjugation and taking subgroups, then the *universal space for G with isotropy in $\mathcal{F}$*, $\mathrm{E}_{\mathcal{F}}G$, is a G-CW complex with the property that $(\mathrm{E}_{\mathcal{F}}G)^H$ is contractible if H is in $\mathcal{F}$ and is empty otherwise. Such spaces are universal for G-actions with isotropy in $\mathcal{F}$, meaning that given any G-CW complex Y whose isotropy groups belong to $\mathcal{F}$, there is a map $Y \to \mathrm{E}_{\mathcal{F}}G$ that is unique up to G-equivariant homotopy equivalence. Thus, $\mathrm{E}_{\mathcal{F}}G$ is unique up to G-equivariant homotopy equivalence.

category vanishes when $X = \mathrm{E}_{\mathcal{VC}yc}G$. (Because of the nice inheritance properties of the assembly map with coefficients, their result proves the K-theoretic Farrell-Jones conjecture for all subgroups of finite products of word hyperbolic groups.) Recently Bartels and Lück have announced the corresponding result in algebraic L-theory. Combining these two theorems proves the Borel conjecture for all subgroups of finite products of word hyperbolic groups.

The ultimate result about the assembly map is that it is an isomorphism. Proving that the obstruction category vanishes achieves this goal, but it is a difficult task. However, with a bit less one can still prove that the assembly map is a split injection using a trick known as *the descent principle*. Using the fact that the continuously controlled assembly map is a map of fixed spectra, one can employ *homotopy fixed point sets* to prove the following theorem (a proof can be found in [8].)

Theorem 4.1 (The Descent Principle)**.** *Let G be a discrete group, $\mathcal{A}$ be an additive G-category and $\mathrm{E}_{\mathcal{F}in}G$ be a finite dimensional G-CW complex. Assume that the K-theory of the category $\mathcal{D}^H(\mathrm{E}_{\mathcal{F}in}G \times [0,1); \mathcal{A})$ vanishes for every finite subgroup H of G.*

Then $A^G_* : H^G_*(\mathrm{E}_{\mathcal{F}in}G; \mathbb{K}^{-\infty}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$ is a split injection.

Consider the case when G is torsion-free. Then the descent principle says that one can prove the assembly map is a split injection by proving that it is an equivalence *unequivariantly.* This is the original version of the descent principle. It was used by Carlsson and Pedersen in [10] to prove the Novikov conjecture for torsion-free groups whose universal space $\mathrm{E}G$ satisfied certain geometric conditions. Since Carlsson and Pedersen's seminal work, the descent principle and continuously controlled algebra have been used to prove split injectivity results (see, for example, [11, 2, 3, 25, 8]). This method was recently used to prove that the assembly map is a split injection for all discrete subgroups of virtually connected Lie groups [8], and for all S-arithmetic subgroups of algebraic groups defined over global fields, regardless of rank, by Ji [19].

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